ON A CONJECTURE FOR HIGHER-ORDER SZEGŐ THEOREMS

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ABSTRACT. We disprove a conjecture of Simon for higher-order Szegő theorems for orthogonal polynomials on the unit circle and propose a modified version of the conjecture.

1. INTRODUCTION

In this paper we investigate probability measures μ supported on the unit circle $\partial\mathbb{D} = \{e^{i\theta} \mid \theta \in [0, 2\pi)\}$. If μ has infinite support, the sequence $1, z, z^2, \dots$ is linearly independent in $L^2(\partial\mathbb{D}, d\mu)$, so Gram–Schmidt orthogonalization provides orthonormal polynomials $\varphi_n(z)$, which obey the recursion relation

$$z\varphi_n(z) = \sqrt{1 - |\alpha_n|^2}\varphi_{n+1}(z) + \bar{\alpha}_n\varphi_n^*(z)$$

with $\varphi_n^*(z) = z^n\overline{\varphi_n(1/\bar{z})}$ and coefficients $\alpha_n \in \mathbb{D}$ called Verblunsky coefficients. Thus, to the measure μ there corresponds the sequence $\alpha = \{\alpha_n\}_{n=0}^\infty \in \mathbb{D}^\infty$. This is, by Verblunsky's theorem [13], a bijective correspondence; see [9, 10] for more information.

With respect to Lebesgue measure on $\partial\mathbb{D}$, μ can be decomposed into an absolutely continuous and a singular part,

$$d\mu = w(\theta)\frac{d\theta}{2\pi} + d\mu_{\mathrm{s}}.$$

The celebrated Szegő theorem for the unit circle (due in this generality to Verblunsky [14]) states that $\alpha \in \ell^2$ is equivalent to

$$\int \log w(\theta)\frac{d\theta}{2\pi} > -\infty. \tag{1.1}$$

Note that, since $\log w \le w-1$, (1.1) is equivalent to $\log w \in L^1(\partial\mathbb{D}, \frac{d\theta}{2\pi})$. This theorem has led the way for many related results on orthogonal polynomials and Schrödinger operators; see [11] for a book-length treatment. In this paper we focus on higher-order Szegő theorems, where (1.1) is replaced by a weaker condition. A conjecture from [9] describes the situation with finitely many singularities. Denote by S the shift operator on sequences,

$$(Sx)_n = x_{n+1}.$$

Date: October 26, 2012.

2010 *Mathematics Subject Classification.* 47B36,42C05,39A70.

Key words and phrases. Szegő theorem.

Conjecture 1.1 ([9, Section 2.8])**.** *Let $m_1, \dots, m_l \in \mathbb{N}$, and let $\theta_1, \dots, \theta_l$ be distinct elements of $[0, 2\pi)$. Then*

$$\int \prod_{k=1}^{l} (1 - \cos(\theta - \theta_k))^{m_k} \log w(\theta) \frac{d\theta}{2\pi} > -\infty \tag{1.2}$$

is equivalent to

$$\prod_{k=1}^{l} (S - e^{-i\theta_k})^{m_k} \alpha \in \ell^2 \quad \text{and} \quad \alpha \in \ell^{2 \max_k (m_k) + 2}. \tag{1.3}$$

The conjecture originated with Simon's proof [9, Section 2.8] for the case $\sum_{k=1}^{l} m_k = 1$; Simon–Zlatoš [12] proved the case $\sum_{k=1}^{l} m_k = 2$, and Golinskii–Zlatoš [2] proved the equivalence under the assumption $\alpha \in \ell^4$. However, we will see that it is not true in general.

In Theorem 2.1, we show that (1.3) is equivalent to the existence of sequences $\beta^{(1)}, \dots, \beta^{(l)}$ such that

$$\alpha = \beta^{(1)} + \dots + \beta^{(l)} \tag{1.4}$$

and, for all k,

$$(S - e^{-i\theta_k})^{m_k} \beta^{(k)} \in \ell^2 \tag{1.5}$$

$$\beta^{(k)} \in \ell^{2 \max_j (m_j) + 2} \tag{1.6}$$

We find this decomposition natural because for each critical point $e^{i\theta_k}$ in (1.2), there is a sequence $\beta^{(k)}$ which corresponds to this critical point through (1.5). However, the conjectured equivalence of (1.2) and (1.4)–(1.6) implies that the degree m_k associated with $e^{i\theta_k}$ affects not only the conditions on the corresponding $\beta^{(k)}$, but all on the $\beta^{(j)}$ through the max in (1.6). We find it more natural to replace (1.6) by

$$\beta^{(k)} \in \ell^{2m_k + 2}. \tag{1.7}$$

This suggests a modified version of the conjecture.

Conjecture 1.2. *Let $m_1, \dots, m_l \in \mathbb{N}$, and let $\theta_1, \dots, \theta_l$ be distinct elements of $[0, 2\pi)$. Then (1.2) is equivalent to the existence of sequences $\beta^{(1)}, \dots, \beta^{(l)}$ such that (1.4), (1.5), (1.7) hold.*

Although this conjecture is incompatible with Conjecture 1.1, they coincide in all the cases which have been proved. However, in the very next case one would naturally proceed to verify, the conjectures differ. For the condition

$$\int (1 - \cos\theta)^2 (1 + \cos\theta) \log w(\theta) \frac{d\theta}{2\pi} > -\infty, \tag{1.8}$$

Conjecture 1.1 predicts necessary and sufficient conditions

$$(S - 1)^2 (S + 1) \alpha \in \ell^2, \quad \alpha \in \ell^6.$$

Conjecture 1.2 is stated in terms of β's, but as we will explain in Section 2, it can be restated in terms of α's and predicts that (1.8) is equivalent to

$$(S-1)^2(S+1)\alpha \in \ell^2, \quad \alpha \in \ell^6, \quad (S-1)^2\alpha \in \ell^4. \tag{1.9}$$

Due to the prohibitive nature of the calculations involved, we only prove a special case.

Theorem 1.3. *Let $(S-1)(S+1)\alpha \in \ell^2$ and $\alpha \in \ell^6$. Then (1.8) is equivalent to $(S-1)^2\alpha \in \ell^4$.*

In particular, Theorem 1.3 disproves Conjecture 1.1:

Corollary 1.4. *Let the Verblunsky coefficients of the measure μ be given by*

$$\alpha_n = \frac{1+(-1)^n}{3(n+1)^{1/4}}. \tag{1.10}$$

Then $(S-1)^2(S+1)\alpha \in \ell^2$ and $\alpha \in \ell^6$, but

$$\int (1-\cos\theta)^2(1+\cos\theta)\log w(\theta)\frac{d\theta}{2\pi} = -\infty. \tag{1.11}$$

Proof. It is straightforward to verify $\alpha \in \ell^6$ and $(S-1)(S+1)\alpha \in \ell^2$; the latter also implies $(S-1)^2(S+1)\alpha \in \ell^2$. However, $(S-1)^2\alpha \notin \ell^4$, so Theorem 1.3 implies (1.11). □

Necessary and sufficient conditions for (1.2) in terms of Verblunsky coefficients have been proved by Denisov–Kupin [1], following work of Nazarov–Peherstorfer–Volberg–Yuditskii [8] for Jacobi matrices. However, these conditions are in a more complicated form which hasn't been successfully related to conditions such as those discussed here. For Jacobi matrices, analogs of the cases $\sum_{k=1}^{l} m_k = 1, 2$ have been proved by Laptev–Naboko–Safronov [5] and Kupin [4].

We end on a pessimistic note: even though all the existing results, including Theorem 1.3, are compatible with Conjecture 1.2, we are not confident that it is true, either. In [7], we analyzed Verblunsky coefficients of the form (1.4), with (1.5) replaced by the stronger condition $(S - e^{-i\theta_k})\beta^{(k)} \in \ell^1$, and with $\alpha \in \ell^p$ for some $p < \infty$. There, the measure is purely absolutely continuous except on an explicit finite set of points. However, this set of possible pure points increases with increasing p, and can contain points not in $\{e^{i\theta_k} \mid k = 1, \dots, K\}$ if $p > 3$. The possibility of these points was shown by Krüger [3] and Lukic [6]. If the analogous phenomenon is true here, it would mean that for large enough m_k, the measure may have points outside of $\{e^{i\theta_k} \mid k = 1, \dots, K\}$ where $\log w$ is not locally L^1, so (1.2) would be false.

2. Decomposition

If $P_1, \dots, P_l \in \mathbb{C}[x]$ are pairwise coprime, then there exist polynomials $U_1, \dots, U_l \in \mathbb{C}[x]$ such that

$$\sum_{j=1}^{l} U_j \prod_{i\neq j} P_i = 1. \tag{2.1}$$

This is easily proved by induction on l, since $\mathbb{C}[x]$ is a principal ideal domain.

Theorem 2.1. *Fix $2 \le p < \infty$. Let $P_1, \dots, P_l \in \mathbb{C}[x]$ be pairwise coprime and $U_1, \dots, U_l \in \mathbb{C}[x]$ be such that (2.1) holds. Then the following are equivalent:*

(i) $\alpha \in \ell^p$ and $P_1(S) \cdots P_l(S)\alpha \in \ell^2$;
(ii) if we define $\beta^{(j)} = U_j(S)\prod_{i\neq j} P_i(S)\alpha$ for $j = 1, \dots, l$, then $\beta^{(j)} \in \ell^p$, $P_j(S)\beta^{(j)} \in \ell^2$ and

$$\alpha = \beta^{(1)} + \cdots + \beta^{(l)} \tag{2.2}$$

(iii) there exist sequences $\beta^{(1)}, \dots, \beta^{(l)} \in \ell^p$ such that (2.2) holds and that $P_j(S)\beta^{(j)} \in \ell^2$ for $j = 1, \dots, l$.

Proof. This proof repeatedly uses the following obvious fact: if $\gamma \in \ell^n$, then $Q(S)\gamma \in \ell^n$, for an arbitrary polynomial $Q \in \mathbb{C}[x]$.

(i) implies (ii): $P_j(S)\beta^{(j)} = U_j(S)P_1(S)\cdots P_l(S)\alpha \in \ell^2$, and $\alpha \in \ell^p$ implies $\beta^{(j)} \in \ell^p$. Finally, (2.1) implies (2.2).

(ii) implies (iii) trivially.

(iii) implies (i): since $\beta^{(1)}, \dots, \beta^{(l)}$ are in ℓ^p, so is their sum α. Further, $P_j(S)\beta^{(j)} \in \ell^2$ implies $P_1(S)\cdots P_l(S)\beta^{(j)} \in \ell^2$, and summing in j, $P_1(S)\cdots P_l(S)\alpha \in \ell^2$. □

For example, in the case considered by Theorem 1.3, take $P_1(z) = (z-1)^2$, $P_2(z) = z+1$. Note that

$$\tfrac{1}{4}(z-1)^2 - \tfrac{1}{4}(z-3)(z+1) = 1,$$

so take $U_1(z) = -\frac{1}{4}(z-3)$, $U_2(z) = \frac{1}{4}$. Then

$$\beta^{(1)} = -\frac{1}{4}(S-3)(S+1)\alpha, \qquad \beta^{(2)} = \frac{1}{4}(S-1)^2\alpha.$$

Notice that $(S-1)^2(S+1)\alpha \in \ell^2$ is equivalent to $(S-1)^2\beta^{(1)}, (S+1)\beta^{(2)} \in \ell^2$. Further, $\beta^{(1)} \in \ell^6$ and $\beta^{(2)} \in \ell^4$ is equivalent to $\alpha \in \ell^6$ and $\beta^{(2)} \in \ell^4$. Thus, Conjecture 1.2 predicts that (1.8) is equivalent to (1.9).

3. Proof of Theorem 1.3

Our proof of Theorem 1.3 follows the method of Simon [9, Section 2.8], Simon–Zlatoš [12] and Golinskii–Zlatoš [2]. Define

$$Z(\mu) = \int (1-\cos\theta)^2(1+\cos\theta)\log w(\theta)\frac{d\theta}{2\pi}. \tag{3.1}$$

Let us assume for a moment that $\alpha \in \ell^2$. Then $\log w \in L^1$ by Szegő's theorem; denote by w_m the moments of $\log w(\theta)$,

$$w_m = \int e^{-im\theta} \log w(\theta) \frac{d\theta}{2\pi},$$

noting that $w_{-m} = \bar{w}_m$. The first few moments are computed in [2],

$$w_0 = \sum_k \log \rho_k^2$$

$$w_1 = -\sum_k \alpha_k \bar{\alpha}_{k-1}$$

$$w_2 = \sum_k \big(-\alpha_k \bar{\alpha}_{k-2} \rho_{k-1}^2 + \tfrac{1}{2} \alpha_k^2 \bar{\alpha}_{k-1}^2\big)$$

$$w_3 = \sum_k \big(-\alpha_k \bar{\alpha}_{k-3} \rho_{k-1}^2 \rho_{k-2}^2 + \alpha_k^2 \bar{\alpha}_{k-1} \bar{\alpha}_{k-2} \rho_{k-1}^2 + \alpha_k \alpha_{k-1} \bar{\alpha}_{k-2}^2 \rho_{k-1}^2 - \tfrac{1}{3} \alpha_k^3 \bar{\alpha}_{k-1}^3\big)$$

This uses the convention $\alpha_{-1} = -1$ and $\alpha_k = 0$ for $k \le -2$, and $\rho_k = \sqrt{1 - |\alpha_k|^2}$. Since

$$(1-\cos\theta)^2(1+\cos\theta) = \tfrac{1}{8}(4 - e^{i\theta} - e^{-i\theta} - 2e^{2i\theta} - 2e^{-2i\theta} + e^{3i\theta} + e^{-3i\theta}),$$

(3.1) implies

$$Z(\mu) = \tfrac{1}{4} \operatorname{Re}(2w_0 - w_1 - 2w_2 + w_3).$$

Thus, for measures with $\alpha \in \ell^2$,

$$Z(\mu) = \tfrac{1}{4} \sum_k \operatorname{Re}\Big(2 \log \rho_k^2 + \alpha_k \bar{\alpha}_{k-1} + 2\alpha_k \bar{\alpha}_{k-2} \rho_{k-1}^2 - \alpha_k^2 \bar{\alpha}_{k-1}^2 - \alpha_k \bar{\alpha}_{k-3} \rho_{k-1}^2 \rho_{k-2}^2 + \alpha_k^2 \bar{\alpha}_{k-1} \bar{\alpha}_{k-2} \rho_{k-1}^2 + \alpha_k \alpha_{k-1} \bar{\alpha}_{k-2}^2 \rho_{k-1}^2 - \tfrac{1}{3} \alpha_k^3 \bar{\alpha}_{k-1}^3\Big) \tag{3.2}$$

Now let μ be arbitrary. Let μ_n be the measure with Verblunsky coefficients

$$\alpha^{(n)} = (\alpha_0, \alpha_1, \dots, \alpha_{n-1}, 0, 0, \dots).$$

The μ_n are known as Bernstein–Szegő approximations of μ, and μ_n converge weakly to μ. It is proved in [2] that

$$Z(\mu) = \lim_{n\to\infty} Z(\mu_n),$$

or equivalently, that the formula (3.2) holds for μ as well (since $Z(\mu_n)$ is just the partial sum for the right-hand side of (3.2)).

To rewrite (3.2) in a more useful way, take

$$L_k = 2\log(1 - |\alpha_k|^2) + 2|\alpha_k|^2 + |\alpha_k|^4$$

$$E_k = -\tfrac{1}{2}(1 - |\alpha_{k-1}|^2 - |\alpha_{k-2}|^2)|\alpha_k - \alpha_{k-1} - \alpha_{k-2} + \alpha_{k-3}|^2$$

$$G_k = -\tfrac{1}{32}|\alpha_k - 2\alpha_{k-1} + \alpha_{k-2}|^4$$

$$J_k = \Big(\tfrac{3}{4}\alpha_k \bar{\alpha}_{k-1} \bar{\alpha}_{k-2} + \tfrac{5}{4}|\alpha_{k-2}|^2 \bar{\alpha}_{k-1} + \tfrac{9}{8}|\alpha_k|^2 \bar{\alpha}_{k-2} + \tfrac{1}{2}\bar{\alpha}_{k-1}^2 \alpha_{k-2} + \bar{\alpha}_{k-2}|\alpha_{k-1}|^2 + \tfrac{23}{16}|\alpha_{k-2}|^2 \bar{\alpha}_{k-2} - \tfrac{5}{4}|\alpha_k|^2 \bar{\alpha}_{k-1} - \tfrac{3}{4}\bar{\alpha}_k \bar{\alpha}_{k-1} \alpha_{k-2} + \tfrac{1}{16}\alpha_k \bar{\alpha}_{k-2}^2 + \tfrac{1}{4}|\alpha_{k-2}|^2 \bar{\alpha}_k\Big)$$

$$H_k = (\alpha_k - \alpha_{k-2})J_k$$

$$F_k = -\alpha_k\bar\alpha_{k-3}|\alpha_{k-1}|^2|\alpha_{k-2}|^2 - \alpha_k^2\bar\alpha_{k-1}\bar\alpha_{k-2}|\alpha_{k-1}|^2 - \alpha_k\alpha_{k-1}\bar\alpha_{k-2}^2|\alpha_{k-1}|^2 - \tfrac13\alpha_k^3\bar\alpha_{k-1}^3$$

$$\begin{aligned} I_k = \Big(&-\tfrac32|\alpha_k|^2 - |\alpha_{k-1}|^2 - \tfrac12|\alpha_{k-2}|^2 + \alpha_k\bar\alpha_{k-2} + \alpha_{k-1}\bar\alpha_{k-2} + \tfrac12|\alpha_k|^2|\alpha_{k-2}|^2 \\ &- \tfrac{31}{32}|\alpha_k|^4 - \tfrac{31}{32}|\alpha_{k-1}|^4 - \tfrac34\alpha_k^2\bar\alpha_{k-1}^2 + |\alpha_k|^2\alpha_k\bar\alpha_{k-1} - |\alpha_{k-1}|^2\alpha_{k-1}\bar\alpha_{k-2} \\ &- |\alpha_{k-1}|^2\alpha_k\bar\alpha_{k-2} - |\alpha_k|^2\alpha_k\bar\alpha_{k-2} - |\alpha_k|^2\alpha_{k-1}\bar\alpha_{k-2} + \tfrac12|\alpha_{k-1}|^2|\alpha_{k-2}|^2\Big) \end{aligned}$$

Lemma 3.1. *Let $\alpha \in \ell^6$ and $(S^2-1)\alpha \in \ell^2$. Then $\{L_k\}, \{E_k\}, \{H_k\}, \{F_k\} \in \ell^1$.*

Proof. $(S^2-1)\alpha \in \ell^2$ implies $(S^3 - S^2 - S + 1)\alpha = (S-1)(S^2-1)\alpha \in \ell^2$. Thus, $\{E_k\} \in \ell^1$, since $|\alpha_k| < 1$ for all k.

$\alpha \in \ell^6$ implies $\{J_k\} \in \ell^2$, so together with $(S^2-1)\alpha \in \ell^2$ it implies $\{H_k\} \in \ell^1$. $\alpha \in \ell^6$ also implies $\{F_k\} \in \ell^1$.

$\alpha \in \ell^6$ implies that $|\alpha_k| < \frac12$ for all but finitely many k. For $z \in [0, \frac14]$, we have the uniform estimate

$$\left|\log(1-z) + z + \tfrac12 z^2\right| \le Cz^3$$

for some finite C. Take $z = |\alpha_k|^2$ to conclude that $|L_k| \le 2C|\alpha_k|^6$ for all but finitely many k; thus, $\alpha \in \ell^6$ implies $\{L_k\} \in \ell^1$. □

A straightforward calculation shows that

$$Z(\mu) = \tfrac14 \sum_k \operatorname{Re}\Big(L_k + E_k + G_k + H_k + F_k + I_k - I_{k-1}\Big). \tag{3.3}$$

However, $\sum_k (I_k - I_{k-1}) = 0$, since it is a telescoping sum and $\lim_{k\to\pm\infty} I_k = 0$. By (3.3) and Lemma 3.1, if $\alpha \in \ell^6$ and $(S^2-1)\alpha \in \ell^2$, then

$$Z(\mu) = C + \tfrac14 \sum_k \operatorname{Re} G_k,$$

where $C = \frac14 \sum_k (L_k + E_k + H_k + F_k)$ is finite. Since $G_k \le 0$, we conclude that $Z(\mu) > -\infty$ is equivalent to $\sum_k G_k > -\infty$, i.e., to $(S^2 - 2S + 1)\alpha \in \ell^4$. This completes the proof of Theorem 1.3.

References

1. S. Denisov and S. Kupin, *Asymptotics of the orthogonal polynomials for the Szegő class with a polynomial weight*, J. Approx. Theory **139** (2006), no. 1-2, 8–28. MR 2220031 (2007a:41050)
2. Leonid Golinskii and Andrej Zlatoš, *Coefficients of orthogonal polynomials on the unit circle and higher-order Szegő theorems*, Constr. Approx. **26** (2007), no. 3, 361–382. MR 2335688 (2008k:42080)
3. Helge Krüger, *On the existence of embedded eigenvalues*, preprint.
4. S. Kupin, *On a spectral property of Jacobi matrices*, Proc. Amer. Math. Soc. **132** (2004), no. 5, 1377–1383. MR 2053342 (2004m:47063)
5. A. Laptev, S. Naboko, and O. Safronov, *On new relations between spectral properties of Jacobi matrices and their coefficients*, Comm. Math. Phys. **241** (2003), no. 1, 91–110. MR 2013753 (2004g:47041)

6. Milivoje Lukic, *Schrödinger operators with slowly decaying Wigner–von Neumann type potentials*, to appear in Journal of Spectral Theory, arXiv:1201.4840.
7. ______, *Orthogonal polynomials with recursion coefficients of generalized bounded variation*, Comm. Math. Phys. **306** (2011), no. 2, 485–509. MR 2824479 (2012f:42048)
8. F. Nazarov, F. Peherstorfer, A. Volberg, and P. Yuditskii, *On generalized sum rules for Jacobi matrices*, Int. Math. Res. Not. (2005), no. 3, 155–186. MR 2130241 (2006h:47054)
9. Barry Simon, *Orthogonal polynomials on the unit circle. Part 1*, American Mathematical Society Colloquium Publications, vol. 54, American Mathematical Society, Providence, RI, 2005, Classical theory.
10. ______, *Orthogonal polynomials on the unit circle. Part 2*, American Mathematical Society Colloquium Publications, vol. 54, American Mathematical Society, Providence, RI, 2005, Spectral theory.
11. ______, *Szegő's theorem and its descendants. Spectral theory for L^2 perturbations of orthogonal polynomials*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2011.
12. Barry Simon and Andrej Zlatoš, *Higher-order Szegő theorems with two singular points*, J. Approx. Theory **134** (2005), no. 1, 114–129. MR 2137559 (2005m:42044)
13. S. Verblunsky, *On Positive Harmonic Functions: A Contribution to the Algebra of Fourier Series*, Proc. London Math. Soc. **38** (1935), no. 1, 125–157. MR 1576309
14. ______, *On Positive Harmonic Functions (second paper)*, Proc. London Math. Soc. **40** (1936), 290–320. MR 1575824

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